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On q-analogues of the quantum harmonic oscillator and the quantum group $SU(2)_q$

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Abstract. The quantum group $SU(2)_q$ is discussed by a method analogous to that used by Schwinger to develop the quantum theory of angular momentum. Such theory of the q-analogue of the quantum harmonic oscillator, as is required for this purpose, is developed.

1. Introduction

The quantum Yang-Baxter (QYB) equation is by now known to play a profound role in a variety of diverse problems in theoretical physics. These include exactly soluble models (like the six- and eight-vertex models) in statistical mechanics (Baxter 1982), integrable model field theories (Sklyanin 1980, Kulish and Sklyanin 1980, Kulish and Reshitikhin 1981, de Vega *et al* 1984, de Vega 1987) exact S-matrix theory (Zamolodchikov and Zamolodchikov 1979), two-dimensional field theories involving fields with intermediate statistics (Frohlich 1987), and conformal field theory (Moore and Seiberg, 1988a, b, Frenkel and Jing 1988, Bernard 1988).

Now, just as the Jacobi identity is an associativity condition for a Lie algebra, so does a QYB equation play a similar role for an algebraic structure of a new type that is a generalisation of a Lie algebra. This structure is sometimes described as a q-deformation (i.e. a deformation or modification, cf (1)-(3) below, that involves a parameter q) of a Lie algebra. Mathematically it is a Hopf algebra (Abe 1980), but it is usually referred to loosely as a quantum group. In any context, the representations of the quantum group associated with its QYB equation are evidently of central importance (Pasquier 1988a, b).

The nature, structure and representation of quantum groups have been developed extensively by Drinfeld (1986), Jimbo (1985, 1986, 1987) and Woronowicz (1987a, b, 1988), while the important work of Faddeev and collaborators can be traced from the deep and useful paper of Faddeev (1987). There are in addition many more papers of considerable interest and importance (Babelon 1984, 1988, Rosso 1987, 1988, Verdier 1986-7, Manin 1987).

In this paper, we study the simplest quantum group $SU(2)_q$, the q-deformation of the Lie algebra of SU(2). This has already been studied extensively by Jimbo, Woronowicz and Pasquier as well as Vaksman and Soibelmann (1988), Matsuda *et al* (1988) and by Kirillov and Reshetikhin (1988). Here we wish to generalise to $SU(2)_q$, Schwinger's approach (Schwinger 1951) to the quantum theory of angular momentum. To achieve this, a q-deformation of the quantum harmonic oscillator formalism has

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to be developed. Much of the present paper is devoted to this task. Then the algebra of $SU(2)_q$, and its representations can be realised in terms of the variables of two independent q-deformed harmonic oscillators. Our results involve explicit coordinate or wavefunction representations as well as abstract Hilbert space versions.

The paper is organised as follows. After a brief review in § 2 of the $SU(2)_q$ algebra, we turn in § 3 to developing the theory of the q-deformation of the simple harmonic oscillator, sufficiently to allow the realisation in § 4 of $SU(2)_q$ in terms of two q-deformed oscillator degrees of freedom. In §§ 5 and 6, we return to the oscillator itself.

2. Review of $SU(2)_q$ algebra

The 'quantum group' $SU(2)_q$ of Sklyanin (1982, 1983), Jimbo (1985), Drinfield (1986) and Woronowicz (1987) is a C-algebra of self-adjoint operators J_x, J_y, J_z, described by the relations

$$[J_z, J_\pm] = \pm J_\pm \tag{1}$$

$$[J_+, J_-] = [2J_z] \tag{2}$$

where $J_\pm = J_x \pm \mathrm{i}J_y$, and we have introduced the abbreviation

$$[x] = \frac{q^x - q^{-x}}{q - q^{-1}}. \tag{3}$$

To see that the algebras of Woronowicz (1987) and Sklyanin (1982, 1983) do indeed imply (1)-(3), we refer to Rosso (1987) in the former case, and proceed as follows in the latter. Writing the quadratic algebra of Sklyanin as

$$[S_0, S_3] = 0 \qquad [S_0, S_\pm] = \pm\tanh^2(\tfrac{1}{2}s)(S_\pm S_3 + S_3 S_\pm)$$

$$[S_+, S_-] = 4S_0 S_3 \qquad [S_3, S_\pm] = \pm(S_0 S_\pm + S_\pm S_0)$$

one writes $\boldsymbol{S}^2 = S_1^2 + S_2^2 + S_3^2$ and checks that the operators

$$K_0 = S_0^2 + \boldsymbol{S}^2 \qquad K_1 = \boldsymbol{S}^2 + (\tanh^2 \tfrac{1}{2}s) S_3^2$$

are Casimir operators. However K_0 and K_1 are not independent, since, in any representation, it can be shown that

$$S_0^2 - S_3^2 \tanh^2 \tfrac{1}{2}s = 4 \sinh^2 \tfrac{1}{2}s.$$

Hence, introducing

$$S_0 = 2\sinh(\tfrac{1}{2}s) \cosh sJ_z$$

$$S_3 = 2\cosh(\tfrac{1}{2}s) \sinh sJ_z$$

$$J_\pm = S_\pm/(2\sinh s)$$

allows it to be shown that J_z, $J_\pm$ so defined obey (1) and (2). Equation (3) involves the parameter $q = \mathrm{e}^s$, s real and positive, so that the right-hand side of (2) is also given by $\sinh 2sJ_z/\sinh s$ and approaches $2J_z$ as $s \to 0$. Thus (1) and (2) define a deformation using q of the Lie algebra of SU(2), to which the word quantum is applied with s playing a role loosely like that of Planck's constant.

Jimbo has shown that there exists one representation of (1) and (2) for each j, $j=0, \frac{1}{2}, 1, \ldots$, and it acts in a Hilbert space V_j with basis $|jm\rangle$, $-j \leq m \leq j$, according to

$$J_z|jm\rangle = m|jm\rangle \tag{4}$$

$$J_\pm|jm\rangle = ([j \mp m][j \pm m+1])^{1/2}|jm \pm 1\rangle$$

$$C|jm\rangle = [j+\tfrac{1}{2}]^2|jm\rangle \tag{5}$$

where C is the Casimir operator of $SU(2)_q$

$$C = [J_z+\tfrac{1}{2}]^2 + J_-J_+ = [J_z-\tfrac{1}{2}]^2 + J_+J_-.$$

We wish to describe a realisation for $SU(2)_q$ analogous to that of Schwinger (1951) and Bargmann (1962) for the angular momenta algebra. In other words, we wish to write $\boldsymbol{J}$ in terms of the creation and destruction operators of a pair of independent q-deformed harmonic oscillator degrees of freedom.

3. The q-deformed harmonic oscillator

We set out by considering an operator a and its adjoint a^+, acting in a Hilbert space with basis $|n\rangle$, $n=0, 1, 2, \ldots$, such that

$$a|0\rangle = 0 \qquad |n\rangle = (a^+)^n|0\rangle/([n]!)^{1/2}$$

where

$$[n]! = [n][n-1]\ldots[1].$$

Then we have

$$a^+|n\rangle = [n+1]^{1/2}|n+1\rangle \tag{6}$$

$$a|n\rangle = [n]^{1/2}|n-1\rangle \tag{7}$$

where

$$aa^+ = [N+1] \tag{8}$$

$$a^+a = [N] \tag{9}$$

and the operator N is such that

$$N|n\rangle = n|n\rangle.$$

Also one has

$$[N, a^+] = a^+ \qquad [N, a] = -a$$

$$q^{rN}a^+q^{-rN} = q^ra^+ \tag{10}$$

$$q^{rN}aq^{-rN} = q^{-r}a^+ \tag{11}$$

and can show q^N commutes with a^+a and aa^+. In fact, (8) and (9) yield

$$aa^+ - q^{-1}a^+a = q^N \tag{12}$$

a deformation of some sort of the usual harmonic oscillator commutation realisation. We will show that a structure of the type just postulated does indeed exist by constructing an explicit realisation of it. However, as (12), involving N explicitly, implies, we are not dealing yet with the natural realisation of it. To proceed towards this, we use (11) to write (12) as

$$q^2 a q^{-N} a^+ - q^{-N/2} a^+ a q^{-N/2} = q. \tag{13}$$

Now, if we define b by

$$b = (q - q^{-1})^{1/2} a q^{-N/2} \tag{14}$$

so that

$$b^+ = (q - q^{-1})^{1/2} q^{-N/2} a^+ \tag{15}$$

then (13) becomes

$$q^2 b b^+ - b^+ b = q^2 - 1. \tag{16}$$

It is easy to treat this directly, and essentially as one does the ordinary harmonic oscillator problem, to find states $|n\rangle$, $n = 0, 1, 2, \ldots$ such that $b|0\rangle = 0$

$$b^+|n\rangle = \{n+1\}^{1/2}|n+1\rangle \tag{17}$$

$$b|n\rangle = \{n\}^{1/2}|n-1\rangle \tag{18}$$

where we use a further abbreviation

$$\{x\} = 1 - q^{-2x}.$$

These are of course the same states as before, as (6) and (7) translate directly into (17) and (18) with the aid of (14) and (15). One has also

$$bb^+ = \{N+1\} \qquad b^+b = \{N\}$$

so that (with $q = e^s$)

$$-2sN = \ln(1 - b^+b).$$

4. q-oscillator description of $\mathrm{SU}(2)_q$

Returning to the deformed oscillator construction of $\mathrm{SU}(2)_q$, we introduce two independent realisations of the type described in § 3 involving operators a_i and a_i^+ and related kets $|n_i\rangle$, $i = 1, 2$. We define

$$J_+ = a_1^+ a_2 \qquad J_- = a_2^+ a_1 \tag{19}$$

and find

$$[J_+, J_-] = [2J_z] \tag{2}$$

where

$$2J_z = N_1 - N_2. \tag{20}$$

Using the definitions (19) and (20), and the results of § 3, it is easy to verify that (1) is satisfied, and that we do indeed have a realisation of the algebra $\mathrm{SU}(2)_q$. Further, with

$$n_1 = j + m \qquad n_2 = j - m$$

we define the related realisations of the $|jm\rangle$ basis of $SU(2)_q$ by means of

$$|jm\rangle = |n_1\rangle|n_2\rangle = \frac{(a_1^+)^{j+m}(a_2^+)^{j-m}|0\rangle}{([j+m]![j-m]!)^{1/2}}.$$

Operation on this basis with J_z, $J_\pm$ given by (19) and (20) leads directly, as required, to (4) and (5).

5. Coordinate description of the q-deformed oscillator

We wish also to display a coordinate representation of the q-deformed harmonic oscillator. To realise this, we set out from the definitions

$$b = e^{2x} - e^{x}e^{s\partial} \qquad \partial = \partial/\partial x \tag{21}$$

$$b^+ = e^{-2x} - e^{s\partial}e^{-x}. \tag{22}$$

It is easy to use

$$e^{u\partial}e^{vx} = e^{vx}e^{u\partial}\, e^{uv} \tag{23}$$

to show that (21) and (22) do indeed satisfy (16). The circumstances under which b^+, as given by (22) is indeed the adjoint of b as given by (21) will be attended to later. From (21), we see that $\Phi_0(x)$ such that $(b\Phi_0)(x) = 0$ is

$$\Phi_0(x) = \exp(x^2/(2s) - \tfrac{1}{2}x)$$

and then

$$\Phi_n(x) = (b^+)^n\Phi_0(x)$$

is given by

$$\Phi_n(x) = \Phi_0(x) \sum_{k=0}^{n} \{{}^n_k\}\, e^{-2kx}(-e^{-s})^{n-k} \tag{24}$$

where $\{{}^n_k\} = \{n\}/(\{k\}\{n-k\})$. One infers (24) by inspection of low n, and proves the result by induction using the lemma

$$\begin{Bmatrix} n \\ k-1 \end{Bmatrix} + e^{-2sk}\begin{Bmatrix} n \\ k \end{Bmatrix} = \begin{Bmatrix} n+1 \\ k \end{Bmatrix}.$$

Setting

$$\Psi_n(x) = (\{n\}!)^{-1/2}\Phi_n(x) \tag{25}$$

we obtain the coordinate representation

$$b^+\Psi_n(x) = \{n+1\}^{1/2}\Psi_{n+1}(x) \qquad b\Psi_n(x) = \{n\}^{1/2}\Psi_{n-1}(x).$$

We note that neither the nature of the variable x nor the coordinate representation of the Hilbert space scalar product has been considered at this point.

6. Orthogonality and scalar product for the q-deformed oscillator

The functions Ψ_n are in fact related to a set of polynomials defined on the unit circle—the Rogers-Szëgo polynomials. For properties of these, we refer to the work of Szëgo (1982), the book of Andrews (1976) and the article of Andrews and Onofri (1984) which has some overlap with §§ 4 and 6 of this paper.

Define the Rogers-Szëgo polynomials by

$$G_n(\xi) = \sum_{k=0}^{n} \begin{Bmatrix} n \\ k \end{Bmatrix} \xi^k.$$

Then the related normalised polynomials

$$\psi_n(z) = (\{n\}!)^{-1/2}(-q^{-1})^n G_n(-zq) \tag{26}$$

where $z = e^{i\theta}$ satisfy

$$(\psi_n, \psi_m) = \delta_{nm}$$

using the scalar product

$$(f, g) = \int_0^{2\pi} d\theta/(2\pi)\mu(\theta) f(z)^* g(z)$$

where $\mu(\theta)$ is the theta function

$$\mu(\theta) = \sum_{N=-\infty}^{\infty} q^{-n^2} z^n.$$

Comparing (26) with (25) and (24), we identify

$$x = -\tfrac{1}{2}i\theta \tag{27}$$

and find

$$\Psi_n(x = -\tfrac{1}{2}i\theta) = \Phi_0(x = -\tfrac{1}{2}\theta)\psi_n(z = e^{i\theta}).$$

Hence

$$\langle n|m\rangle = \int_0^{2\pi} d\theta/(2\pi)\sigma(\theta)\Psi_n(\theta)^*\Psi_m(\theta) \tag{28}$$

where

$$\sigma(\theta) = \mu(\theta) \exp \theta^2/(4s). \tag{29}$$

Using (27), we rewrite (21) and (22) as

$$b = \exp(-i\theta) - \exp(-\tfrac{1}{2}i\theta) \exp(2is\partial/\partial\theta) \tag{30}$$

$$b^+ = \exp(i\theta) - \exp(2is\partial/\partial\theta) \exp(\tfrac{1}{2}i\theta). \tag{31}$$

Superficially, if $i\partial/\partial\theta$ is Hermitian, it seems plausible that b^+ as given by (31) is indeed conjugate to b as given by (30). A more careful proof, using (28) and (29), depends on the evident result

$$\sigma(\theta - 2is) = \sigma(\theta)$$

and completes our discussion.

7. Further discussion

The above results give rise directly to one wavefunction realisation of $SU(2)_q$. It is not the only one available to us. Another, for each fixed $l = 0, \frac{1}{2}, 1, \ldots$, involves

$$J_+ = e^{i\phi}[l + i\partial/\partial\phi] \tag{32}$$

$$J_- = e^{-i\phi}[l - i\partial/\partial\phi] \tag{33}$$

$$J_z = -i\partial/\partial\phi. \tag{34}$$

That (32)-(34) obey (1)-(3) is easily checked directly using (23). Further, use of

$$\langle\phi|jm\rangle = e^{im\phi}([l+m]![l-m]!)^{-1/2} \tag{35}$$

shows that the action of $\boldsymbol{J}$ is in accord with (4) and (5). A suitable representation of the scalar product of the Hilbert space in use here can be given (Sklyanin 1982, 1983), but this will not be gone into here.

There are a variety of directions in which further study is under way. Perhaps the most important one reflects the fact that a deeper view of everything discussed here unquestionably exists. The Bargmann (1962) Hilbert space of the quantum harmonic oscillator, and of angular momentum described (as by Schwinger 1951) in terms of two independent such degrees of freedom, involve respectively one complex number z, two complex numbers z_1 and z_2. The angular integrals in $e^{i\theta}$ and $e^{i\theta_1}$, $e^{i\theta_2}$ control orthogonality, but the radial variables are vital to the natural development of the formalism. We have here the analogues in the q-deformed theory of the $e^{in\theta}$, but not of the radial variables. Some embedding is required of our work in a deeper view. It would show us for example how the theta function in (28) arises naturally, and how to replace the l in (32) and (33) by a differential operator in some new variable, α say, or what is the same, to extend (35) to the q-analogue

$$\langle\alpha\theta|jm\rangle_q$$

of the spherical harmonics.

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